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Low-Temperature Behavior of a One-Dimensional Random Ising Model

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The random Ising chain is a very simple model with a large number of metastable states. Simple analytical calculation of the relaxation of energy and magnetization is presented. The effect of a nonzero magnetic field is discussed qualitatively. The slow relaxation in this simple model resembles that observed in spin glasses. A weak magnetic field can produce rather strong effects. The magnetization is shown to be a nonanalytic function of the field. The field also greatly alters the metastability characteristics.

KEY WORDS: Metastable states; kinks.

I. INTRODUCTION

The existence of a large number of metastable states is responsible for some properties of spin glasses.(1–4) In particular the extremely slow decay of the remanent magnetization and the long relaxation time of the thermal energy at low temperatures are characteristic features of spin glasses. These features have been studied extensively in the literature. They are highly nontrivial and their analyses are generally very involved.

In this paper, we attempt to gain some understanding of the metastability by studying the simplest system with a large number of metastable states, namely, the one-dimensional Ising model with random nearest-neighbor interaction.(1,5,6) The simplicity of this model allows us to carry

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out some simple analytical calculations which are very instructive. However, even for such a simple model, we have not been able to push analytic calculations very far. Numerical calculations have also been done to supplement the analytic work. In the case of a nonzero magnetic field, the problem becomes very difficult.(7–9) We have been able to give only a heuristic analysis, which gives a qualitative account of some very dramatic effect of the field.

The Hamiltonian of the Ising chain is described by

$$H = -\frac{1}{2}\sum_{i=1}^{N} J_i t_i \sigma_i \sigma_{i+1} - h \sum_{i=1}^{N+1} \sigma_i \tag{1.1}$$

where the spin variables $\sigma_i = \pm 1$. The sign of the bond energies t_i are ± 1 with equal probability and are distributed independently. The magnitudes of the bond energies J_i are also independently distributed with certain probability distribution $p(J)$. In this paper general forms of $p(J)$ will be considered, but we confine most of our calculation to the case of exponential distribution

$$p(J) = \exp(-J), \qquad J > 0 \tag{1.2}$$

When the small-J part of $p(J)$ is crucial, we shall consider cases with

$$p(J) \propto J^n, \qquad n > -1 \quad \text{for small } J \tag{1.3}$$

Let us describe some terminologies to be used below. If a bond J_i is greater than its nearest neighbor bonds J_{i-1} and J_{i+1}, J_i will be called an energy maximum, on the other hand, if J_i is smaller than J_{i-1} and J_{i+1}, J_i will be called an energy minimum. A collection of n consecutive bonds (J_{i+1} through J_{i+n}) form an energy well if both J_i and J_{i+n+1} are greater than all of the n bonds. The lowest energy among the n bonds, denoted J_g, is called the bottom of the well. It is clear that each energy minimum is the bottom of a well, and an energy well may contain several energy minima.

The dynamics is assumed to be generated by single spin-flips caused by thermal noises coupled to each spin. For our model, the relevant dynamics can be conveniently described by the motion of kinks over energy barriers. By a kink at the energy bond J_i, we mean that $J_i t_i \sigma_i \sigma_{i+1} < 0$. The energy barrier for a kink to move from the bottom of the well to a lower energy bond outside the well is called the barrier height of the well. We assume that when the observation time t is longer than $e^{\Lambda/T}$, the energy barrier Λ is readily overcome, and when $t < e^{\Lambda/T}$, the barrier is formidable.

In Section 2 we review the static properties of the system at low temperatures and zero field. In this case only a small number of kinks exist in the system. On the other hand, when the system is at a high temperature, or in a large magnetic field, nearly one half of the bonds have kinks.

In Section 3 dynamic effects are considered in the low-temperature limit. From the motion of kinks a simple argument is given to describe the relaxation of the thermal energy and the magnetization, when the system is suddenly quenched to a low temperature, or when the large magnetic field is suddenly switched off. For the $p(J)$ given by Eq. (1.2), the decay of thermal energy and magnetization per spin at $T \ln t \gg 1$ are found to follow

$$\Delta E/N \sim T \ln t \cdot t^{-2T} \tag{1.4}$$

$$M/N \sim t^{-T} \tag{1.5}$$

The magnetization decays more slowly.

In Section 4 time-dependent thermodynamic quantities are studied in more detail. States with even (or odd) numbers of kinks are assumed to be in thermal equilibrium in the energy wells considered. The time-dependent energy so obtained has the leading term agree with the simple argument given in Section 3; and in the infinite-time limit, the energy reduces to the equilibrium value. The energies and the specific heats are also calculated numerically.

In Section 5 the zero-field susceptibility is derived and calculated numerically. At each time, a hump in the susceptibility is observed as a function of temperature.

The effect of a weak magnetic field is studied qualitatively in Section 6. The magnetization at very low temperatures is found to follow a power law of the field. In particular, for $n = 0$ [see Eq. (1.3)], we obtain

$$M/N \sim h^{1/3} \tag{1.6}$$

The specific heat at very low temperatures is found to be proportional to T. For $n = 0$, it is approximately independent of h for $T \ll h^{2/3}$. The effect of h on metastability is discussed.

Further discussion is given in Section 7.

2. STATIC PROPERTIES AT LOW TEMPERATURES

At $T = 0$ and $h = 0$, no kinks exist in the system. The ground-state energy per spin of the Ising chain is

$$E_g/N = -\tfrac{1}{2}\langle J \rangle_p \tag{2.1}$$

where $\langle \ \rangle_p$ is the average over the distribution $p(J)$. For the exponential distribution, Eq. (1.2), $E_g/N = -1/2$. At finite temperature, or finite field, some kinks will be excited. When a kink is excited at the bond J_i, the energy of the system is increased by the amount J_i. In the following, we use ΔE to denote the energy of the system, as measured from E_g. That is, $\Delta E = E - E_g$.

2.1. Thermal Energy and Specific Heat at Zero Field

In thermal equilibrium, the probability of the existence of a kink at an energy bond J_i is independent of the existence of kinks at other bonds. At temperature T, the probability of a kink at J_i is given by

$$f_k(J_i) = \left(1 + e^{J_i/T}\right)^{-1} \tag{2.2}$$

The energy of the system in equilibrium with a heat bath at a temperature T is

$$\Delta E(T) = \left\langle \sum_i J_i \left(1 + e^{J_i/T}\right)^{-1} \right\rangle_p \tag{2.3}$$

At very low temperatures, only energy bonds with $J_i \ll 1$ may have a kink. At $J \ll 1$, assume $p(J)$ has the general form

$$p(J) = AJ^n, \qquad J \ll 1 \tag{2.4}$$

where A is a constant, and higher-order terms are neglected. It is necessary that $n > -1$; otherwise the integration of $p(J)$ diverges.

From Eqs. (2.3) and (2.4)

$$\Delta E(T)/N = AI_{n+1} T^{n+2} + \cdots \tag{2.5}$$

where

$$I_n = \int_0^\infty \frac{z^n \, dz}{1 + e^z} \tag{2.6}$$

Since I_n are related to the Riemann zeta functions $\zeta(n)$, $I_n = n!\,(1 - 2^{-n})\,\zeta(n+1)$, they can be obtained directly from mathematical tables. For example, $I_0 = \ln 2$, $I_1 = \pi^2/12$, etc.

For the exponential distribution, Eq. (1.2), it is easy to calculate the higher-order terms, and obtain

$$\frac{\Delta E(T)}{N} = \sum_{n=0}^{\infty} \frac{(-1)^n I_{n+1} T^{n+2}}{n!} \tag{2.7}$$

It is interesting to note that the leading term in ΔE, Eq. (2.5), is contributed from the kinks at the energy minima alone. The energy of the kinks at the energy minima is given by

$$\frac{\Delta E}{N} = \int_0^\infty J f_k(J) p(J) \left[\int_J^\infty p(J')\, dJ'\right]^2 dJ \tag{2.8}$$

The factor $[\int_J^\infty p(J')\,dJ']^2 = 1 - O(J^{n+1})$ is the probability that the two neighboring bonds of J have energies higher than J. The leading term of Eq. (2.8) is exactly the same as Eq. (2.5), and the first correction term is of order T^{2n+3}.

From the energy of the system, Eq. (2.3), (2.5), or (2.7), the specific heat is readily obtained by $C = \partial E/\partial T$. At low temperatures, Eq. (2.5) gives

$$\frac{C}{N} = A(n+2)I_{n+1}T^{n+1} \tag{2.9}$$

2.2. Magnetic Susceptibility

In the zero-field limit, the susceptibility of a spin system is given by

$$\chi(T, h \to 0) = \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\langle\sigma_i\sigma_j\rangle \tag{2.10}$$

For the Ising chain with nearest-neighbor interaction,

$$\langle\sigma_i\sigma_{i+k}\rangle = \prod_{j=i}^{i+k-1}\tanh\left(\frac{t_iJ_i}{2T}\right) \tag{2.11}$$

Since $t_i = \pm 1$ with equal probability, when we sum over all systems of different t_iJ_i, the thermal average $\langle\sigma_i\sigma_{i+k}\rangle$ vanishes for $k \neq 0$. The only contribution to χ is from $\langle\sigma_i^2\rangle = 1$. Therefore the random Ising chain has $\chi = N/T$ in the zero-field limit. The magnetization in the low-field ($h/T \ll 1$) limit is of the form

$$\frac{M}{N} = \frac{h}{T} + O(h^3) \tag{2.12}$$

This result is independent of the distribution $p(J)$.

3. MOTION OF KINKS AND RELAXATION OF ENERGY AND MAGNETIZATION

At high temperatures all spins are randomly oriented, and the number of kinks in the system is $N/2$. When the system is suddenly quenched to a low temperature, the energy of the system relaxes rapidly first as all kinks move down toward the energy minima and annihilate each other when two kinks meet at a bond. When there are an even number of kinks moving down to an energy minimum, the kinks annihilate and no kinks remain. On the other hand, an energy minimum has one kink left if an odd number of kinks move down to the minimum. Since we are not interested in this rapid relaxation period, we will consider our initial state as that with one or no kink at each energy minimum with probability $1/2$. What we will investigate is the behavior of the system after the rapid decay to this initial state.

3.1. Relaxation of the Thermal Energy

Consider an energy minimum J_n as shown in Fig. 1. J_n is the bottom of an energy well of barrier height Λ. If there is a kink in the well ($f_k = 1/2$ initially), the kink spends most of the time at the minimum J_n. But if there is a lower minimum J_m, then there is a finite probability that the kink climbs over the barrier and moves down to J_m. After the kink has moved to J_m the probability that a kink exists at J_n reduces to zero. If there was a kink ($f_k = 1/2$) at J_m, it will annihilate with the kink from J_n; and if there was no kink at J_m, the kink from J_n will stay at J_m. Therefore the probability that a kink exists at J_m remains unchanged ($= 1/2$).

The characteristic time for a kink to climb over an energy barrier Λ is $t = \tau \exp(\Lambda/T)$. We will assume that the time for a kink to move to a neighboring lower energy bond is $\tau = 1$. This suggests that the probability for a kink to exist in any well of barrier height Λ is $1/2$ for $t < \exp(\Lambda/T)$.

If we define $p(J)D(J,\Lambda)\,dJ$ as the number of energy wells per spin with energy of the bottom lying between J and $J + dJ$, and barrier height larger than Λ, then the energy of the system at time t is

$$\frac{\Delta E(t)}{N} = \frac{1}{2}\int_0^{\infty} Jp(J)D(J,\Lambda)\,dJ \tag{3.1}$$

where $\Lambda = T\ln t$ and $1/2$ is the probability that a kink exists at the bottom of the energy well.

The factor $D(J,\Lambda)$ is given by

$$D(J,\Lambda) = \sum_{n=1}^{\infty} nP^2(J+\Lambda,\infty)P^{n-1}(J,J+\Lambda) \tag{3.2}$$

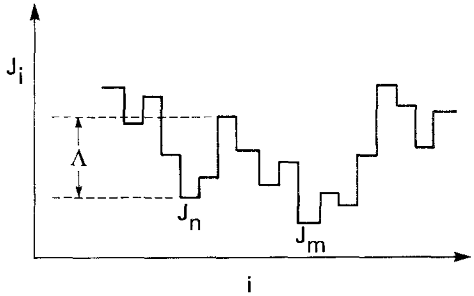

Fig. 1. A J_i vs. i plot. If there is a kink at the energy minimum J_n, then there is a finite probability that the kink climbs over the barrier Λ and moves down to the lower minimum J_m.

where

$$P(a,b)=\int_a^b p(J)\,dJ \tag{3.3}$$

Each term in the summation represents an energy well of n bonds. The second factor $P^2(J+\Lambda,\infty)$ is the probability that we have two energy bonds higher than $J+\Lambda$ to ensure that the barrier of the well is higher than Λ. The last factor $P^{n-1}(J,J+\Lambda)$ is the probability that we have $n-1$ intermediate energy bonds, and the first factor n indicates that the bottom of the well J can occur at any of the n positions.

For the exponential distribution, Eq. (1.2), we get

$$D(J,\Lambda)=e^{-(2J+2\Lambda)}\left[1-e^{-J}(1-e^{-\Lambda})\right]^{-2} \tag{3.4}$$

and

$$\Delta E/N=\sum_{n=1}^{\infty}\frac{n}{2(n+2)^2}\left(1-e^{-\Lambda}\right)^{n-1}e^{-2\Lambda} \tag{3.5}$$

At $\Lambda \ll 1$ we have a logarithmic decay

$$\begin{aligned}\frac{\Delta E}{N}&=\frac{1}{18}\left[1-\frac{7}{8}\Lambda+\cdots\right]\\&=\frac{1}{18}\left[1-\frac{7}{8}T\ln t+\cdots\right]\end{aligned} \tag{3.6}$$

and for $\Lambda \gg 1$ a power law decay with exponent $2T$

$$\begin{aligned}\frac{\Delta E}{N}&\simeq\frac{\Lambda}{2}e^{-2\Lambda}\\&=\frac{1}{2}T\ln t\cdot(t)^{-2T}\end{aligned} \tag{3.7}$$

The behavior of ΔE at $\Lambda \gg 1$ depends strongly on the distribution $p(J)$. For example, it would decay much faster if $p(J)$ is a Gaussian distribution.

We have compared Eq. (3.5) with computer simulation results (to 3000 Monte Carlo Steps per spin) for systems of 1000 spins. In the simulation the time is set equal to 1 when the energy of the system is equal to 1/18. For $T=0.01$, Eq. (3.5) is in good agreement with the Monte Carlo result. For $T=0.1$ the time-dependent energy predicted by Eq. (3.5) (solid line in Fig. 2) is lower than the Monte Carlo result (dashed line in Fig. 2). This is expected because our simple theory assumes that all kinks vanish as $t\to\infty$, i.e., the equilibrium thermal energy is neglected. For $T=0.01$ the equilibrium thermal energy $\Delta E_{\mathrm{eq}}/N=8.0\times10^{-5}$ [obtained from Eq. (2.7)] is in

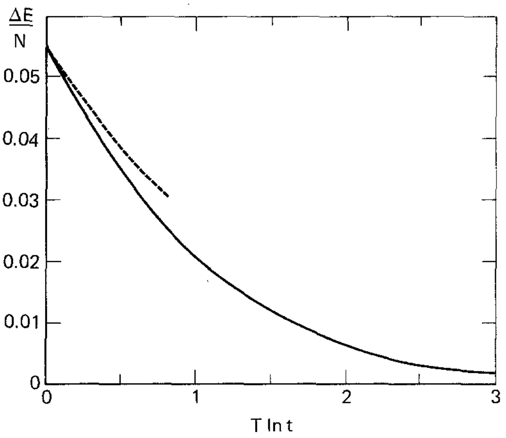

Fig. 2. Relaxation of the thermal energy when the system is quenched from a high temperature to a low temperature T. The solid line is obtained by assuming that at most one kink can exist in a well, Eq. (3.5). The dashed line is the Monte Carlo simulation for systems of 10^3 spins (averaged over five systems) at $T = 0.1$.

fact negligible. While for $T = 0.1$, $\Delta E_{\text{eq}}/N = 6.7 \times 10^{-3}$ is comparable to the thermal energy at $\Lambda = 0$.

3.2. Relaxation of Remanent Magnetization

When the Ising chain is exposed to a large magnetic field all spins are parallel to the external field. All antiferromagnetic bonds have a kink. When the magnetic field is suddenly switched off, and the system is kept at a low temperature, the kinks move down to the energy minima and annihilate each other as described in Section 3. As before, we will consider our initial state as the one with a kink at each energy minimum with probability 1/2. The relaxation of energy is the same as described in Section 3.1.

As pointed out by Fernández and Medina,$^{(5)}$ during the motion of kinks, a spin flips each time when passed by a kink. If a spin is passed by an even number of kinks, its direction does not change. On the other hand, if the spin is passed by an odd number of kinks, its spin direction is reversed. Therefore, all spins which have the probability to be passed by kinks do not contribute to the magnetization when we average over $t_i = \pm 1$.

During the fast relaxation, all kinks move down to energy minima. The only spins that will not be passed by any kinks are the pair of spins which are joined by ferromagnetic energy maxima. Since each energy bond has a probability 1/2 to be ferromagnetic, and a pair of spins has a net magnetic moment of 2 units, the remanent magnetization of our initial state is equal to the number of energy maxima in the system. The number is $N/3$ because if we generate three numbers independently, the probability that the second number is the largest is $1/3$, independent of the distribution $p(J)$.

After the fast relaxation, we use the assumption made in Section 3.1 that kinks in the bottom of the energy wells can climb over the barrier, cross the energy maximum, and move into the neighboring well. At time t consider the spins and the energy bonds located between the bottoms of two adjacent energy wells of barrier higher than $T\ln t$. The only spins that will not be passed by any kink are the pair of spins joined by the bond which has the largest value of J. This energy bond is further required to be ferromagnetic.

Therefore the magnetization of the system at time t is equal to the number of energy wells with barrier height larger than $\Lambda = T\ln t$.

$$\frac{M(t)}{N} = \int_0^\infty p(J)D(J,\Lambda)\,dJ \tag{3.8}$$

For the exponential distribution, we get

$$\frac{M(t)}{N} = \eta + \frac{2\eta^2}{(1-\eta)^3}\left[-\Lambda + (1-\eta) + \frac{1}{2}(1-\eta)^2\right] \tag{3.9}$$

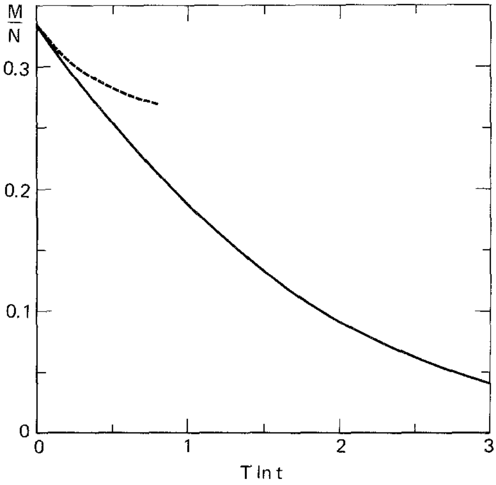

Fig. 3. Relaxation of remanent mangetization when a strong magnetic field is suddenly turned off at a low temperature T. The solid line is obtained from the assumption that at most one kink can exist in an energy well, Eq. (3.9). The dashed line is the Monte Carlo result for five systems of 10^3 spins, generated by the exponential distribution, Eq. (1.2), at $T = 0.1$.

where $\eta = e^{-\Lambda}$. At $\Lambda \ll 1$,

$$\frac{M(t)}{N} = \frac{1}{3}\left(1 - \frac{\Lambda}{2} + \cdots\right) = \frac{1}{3}\left(1 - \frac{1}{2} T \ln t + \cdots\right) \tag{3.10}$$

and for $\Lambda \gg 1$

$$\frac{M(t)}{N} \simeq e^{-\Lambda} = (t)^{-T} \tag{3.11}$$

The relaxation of the magnetization is much slower than the relaxation of thermal energy. This result is in agreement with Monte Carlo observation. The quantitative agreement between Eq. (3.9) and the Monte Carlo data, however, is not good. (See Fig. 3.)

4. TIME-DEPENDENT THERMODYNAMIC PROPERTIES

In the previous section we have assumed that: (i) For any energy well with barrier height Λ the probability to have a kink is $1/2$ for $t < e^{\Lambda/T}$ and 0 otherwise. (ii) The kinks exist only at the bottom of the energy well. The assumption that the probability changes abruptly is incorrect because it should vary smoothly. But when we sum over energy wells of various Λ the discontinuity of the probability does not make any significant error as long as the distribution function $p(J)$ does not have a sharp change. The assumption that only one kink can exist in the energy well does make our energy smaller than the Monte Carlo result.

To improve our theory we modify our assumption as follows: (i) At time t only energy wells with barrier height larger than $\Lambda = T \ln t$ need to be considered. For mathematical convenience we ignore energy bonds which are higher than $\Lambda + J_g$ (J_g is the energy of the bottom bond) in each well, because these bonds are not likely to have a kink at low temperatures. (ii) Any energy bond considered may have a kink. Since the kinks can be created or annihilated in pairs within a well, the energy well may have either even numbers of kinks or odd numbers of kinks, with equal probability $1/2$. (iii) If the number of kinks in a well is even, the well can be in any spin state with even number of kinks (will be called even state hereafter). The probability of the well to be in a state $\{\sigma_i\}$ is proportional to the Boltzmann factor $\exp(-H\{\sigma_i\}/T)$. Similarly, all odd states may exist in a well with probability proportional to the Boltzmann factor. However, odd states and even states do not mix in any energy well.

4.1. Time-Dependent Thermodynamic Quantities

With these assumptions we can define a time-dependent free energy $F(\Lambda)$ in the following way. For an energy well of n bonds it can be shown that

$$Z_E\{J_i\} \equiv \sum_{\text{even states}} \exp(-H\{\sigma_i\}/T) \tag{4.1}$$

$$= \tfrac{1}{2}(Z_+ + Z_-) \tag{4.2}$$

and

$$Z_O\{J_i\} \equiv \sum_{\text{odd states}} \exp(-H\{\sigma_i\}/T) \tag{4.3}$$

$$= \tfrac{1}{2}(Z_+ - Z_-) \tag{4.4}$$

where Z_+ and Z_- are defined as

$$Z_\pm \equiv e^{-E_g/T} \prod_{i=1}^{n} (1 \pm e^{-J_i/T}) \tag{4.5}$$

The well may have an even or odd number of kinks with probability 1/2. The average free energy of the well is then

$$\begin{aligned} F\{J_i\} &\equiv \tfrac{1}{2}F_E\{J_i\} + \tfrac{1}{2}F_O\{J_i\} \\ &= -\tfrac{1}{2}T(\ln Z_E + \ln Z_O) \end{aligned} \tag{4.6}$$

Other thermodynamic quantities of the energy well are also well defined and consist of an even part and an odd part.

For any finite Ising chain with fixed boundaries, or with periodic boundary conditions, the number of kinks in the system is even (or odd) all the time. The equilibrium free energy of the system is simply F_E (or F_O). And for a free boundary Ising chain with $N \gg 1$, $Z_- \ll Z_+$, both F_E and F_O are equal to the equilibrium free energy.

The time-dependent thermodynamic quantity A of the Ising chain at time t is then defined as the summation of quantities $A\{J_i\}$ of all energy wells with barrier height larger than $\Lambda = T\ln t$.

$$\begin{aligned} \frac{A(t)}{N} = \sum_{n=1}^{\infty} n \int_0^\infty p(J_g)\, dJ_g\, P^2(J_g + \Lambda, \infty) \\ \times \int_{J_g}^{J_g+\Lambda} p(J_2)\, dJ_2 \int_{J_g}^{J_g+\Lambda} p(J_3)\, dJ_3 \cdots \\ \times \int_{J_g}^{J_g+\Lambda} p(J_n)\, dJ_n \tfrac{1}{2}(A_E\{J_i\} + A_O\{J_i\}) \end{aligned} \tag{4.7}$$

Here J_g is the bottom of a well of n bonds. The $(n-1)$ intermediate energies $J_2, J_3, \ldots, J_n$, lie between J_g and $J_g + \Lambda$. $A_{E,O}\{J_i\}$ is the quantity A of the well of energies $\{J_g, J_2, J_3, \ldots, J_n\}$ in even (or odd) states. Other factors are the same as given in Eq. (3.2).

4.2. T^2 Term of the Time-Dependent Energy

The leading contribution to the time-dependent energy comes from the odd term $E_O\{J_i\}$ and with only one kink in a well:

$$E_O\{J_i\} = \frac{\sum_i J_i e^{-J_i/T} + \text{3-kink terms} + \cdots}{\sum_i e^{-J_i/T} + \text{3-kink terms} + \cdots}$$

$$= J_g + \frac{\sum_{i \neq g} \Delta_i e^{-\Delta_i/T}}{1 + \sum_{i \neq g} e^{-\Delta_i/T}} + \cdots \tag{4.8}$$

where J_g is the bottom of the well and $\Delta_i = J_i - J_g$. If we consider only the leading term J_g, Eq. (4.7) then gives the same result as Eq. (3.1).

Since the term J_g does not contain a Boltzmann factor, when the energy is averaged over the distribution $p(J)$ we get a term zeroth order in T. If the integrand contains a factor $e^{-J_i/T}$, each power in J_i and dJ_i contributes a power T when the integration over dJ_i is carried out. Therefore the second term in the time-dependent energy is the T^2 term, contributed from the part which contains only one kind of Boltzmann factor,

$$E^{(2)}\{J_i\} = \sum_{i \neq g} \frac{\Delta_i}{1 + e^{\Delta_i/T}} \tag{4.9}$$

There are $(n-1)$ terms in Eq. (4.9). Each term has the same contribution. Equation (4.7) reduces to

$$\Delta E^{(2)}(t)/N = \sum_{n=1}^{\infty} \frac{n(n-1)}{2} \int_0^{\infty} e^{-(2\Lambda + 3J_g)} dJ_g P^{n-2}(J_g, J_g + \Lambda)$$

$$\times \int_0^{\Lambda} e^{-(J_g + \Delta)} \frac{\Delta d\Delta}{1 + e^{\Delta/T}}$$

$$= \int_0^{\infty} \frac{e^{-4J_g} e^{-2\Lambda}\, dJ_g}{\left[1 - e^{-J_g}(1 - e^{-\Lambda})\right]^3} \cdot \int_0^{\Lambda} e^{-\Delta} \frac{\Delta d\Delta}{1 + e^{\Delta/T}} \tag{4.10}$$

Let $\eta = e^{-\Lambda}$; then the first integration gives

$$I = \left[\tfrac{1}{2} - 3\eta + 3(-\ln \eta)\eta^2 + \tfrac{3}{2}\eta^2 + \eta^3\right]/(1 - \eta)^4 \tag{4.11}$$

The second integration yields $\pi^2 T^2/12 + O(T^3)$ for $\Lambda/T = \ln t \gg 1$. Therefore the T^2 term in the time-dependent energy is

$$\frac{\Delta E^{(2)}(t)}{N} = \frac{\pi^2 T^2}{12} \left[\frac{1}{4} + \frac{\Lambda}{10} + \cdots\right] \quad \text{for} \quad \ln t \gg 1 \quad \text{and} \quad T \ln t \ll 1 \tag{4.12}$$

$$= \frac{\pi^2 T^2}{12} \left[\frac{1}{2} - e^{-\Lambda} + \cdots\right] \quad \text{for} \quad T \ln t \gg 1 \tag{4.13}$$

At $T \ln t \gg 1$, $\Delta E^{(2)}/N$ is only one-half of the equilibrium value. This is expected because when $T \ln t \gg 1$ both $E_E = E_O = \frac{1}{2} E_{\text{eq}}$, and Eq. (4.13) is contributed from E_O only. The leading contribution of even states to the time dependent energy comes from two kink states with one kink at J_g and another kink at any of the $(n-1)$ bonds $J_i \neq J_g$, i.e.,

$$E_E\{J_i\} = \sum_{i \neq g} \frac{\Delta_i + 2J_g}{1 + e^{(\Delta_i + 2J_g)/T}} + \cdots \tag{4.14}$$

For $\ln t \gg 1$ and $T \ln t \ll 1$, Eq. (4.14) contributes a term proportional to T^3 to the time-dependent energy.

At $T \ln t \ll 1$ the contributions to the time-dependent energy and specific heat are mainly from the odd states. When $T \ln t \gg 1$ both even states and odd states should have equal contribution since $Z_- \ll Z_+$ and $Z_E \simeq Z_O$. But it is difficult to show analytically how the contribution of even terms changes from T^3 to T^2.

4.3. Numerical Calculation of Time-Dependent Thermodynamic Quantities

Equation (4.7) cannot be evaluated directly. To determine the time-dependent thermodynamic quantities numerically one can randomly generate a long Ising chain according to $p(J)$ and find out all the energy wells need to be considered at each value of $T \ln t$. The thermodynamic quantities of the Ising chain are the sums of the corresponding quantities of the energy wells.

There is an alternative way to generate the energy wells. We consider an arbitrary position n and generate a set of energies randomly. If J_n comes out to be the bottom of an energy well of desired barrier height, we calculate the thermodynamic properties of the well. Otherwise, we ignore the set of energies and repeat the process for a new position. Since all positions along the chain are equivalent for $N \gg 1$, the number of energy sets generated (including the ones ignored) is equal to the number of spins in the chain. In real calculations we do not have to generate a large set of energies each time. It contains the following steps:

(i) To begin, generate an energy, denoted by J_g, and assume that it is the bottom of an energy well.

(ii) Generate the energy bonds to the right of the bottom, $J_1, J_2, J_3, \ldots$ until an energy less than J_g, or greater than $J_g + \Lambda$, is generated. If an energy less than J_g is generated, J_g is not the bottom of the well. We return to step (i) and repeat the process. If an energy greater than $J_g + \Lambda$ is generated the right-hand side of the well is completed.

(iii) Generate the energy bonds to the left of the bottom, $J_{-1}, J_{-2}, J_{-3}, \ldots$ until an energy less than J_g, or greater than $J_g + \Lambda$, is generated.

If an energy less than J_g is generated, return to step (i). If an energy greater than $J_g + \Lambda$ is generated, the energy well is completed.

(iv) Calculate thermodynamic properties of the energy well from Eq. (4.6). Go to step (i) and generate another energy well.

After a large number (10^3–10^4) of energy wells of various numbers of bonds have been generated, a thermodynamic quantity (per spin) of the Ising chain is presented by the sum of the corresponding quantities of all the wells divided by the total number of J_g generated (including those which fail to complete a well). For a given value of Λ the ratio of the number of energy wells completed to the number of J_g generated is equal to $\int_0^\infty p(J)D(J,\Lambda)\,dJ$. In our calculation this ratio is obeyed when more than 10^3 wells are generated.

The energy and the specific heat for $T = 0.1$ and 0.01, respectively, are shown in Figs. 4–6. Each data point represents an average over 10^3 energy wells. At $T = 0.01$ the energy is contributed mainly from the kinks at the bottoms of the energy wells. The energy calculated from the simulated energy wells is the same as Eq. (3.5) (solid line in Fig. 4). For $T = 0.1$, the numerically calculated energy is higher than Eq. (3.5). The difference comes from the kinks not at the bottoms of wells.

From the way we define the time-dependent thermodynamic quantities, the specific heat is related to the energy by

$$C(T,\Lambda) = \partial E(T,\Lambda)/\partial T \tag{4.15}$$

An alternative way to define the time-dependent specific heat is discussed

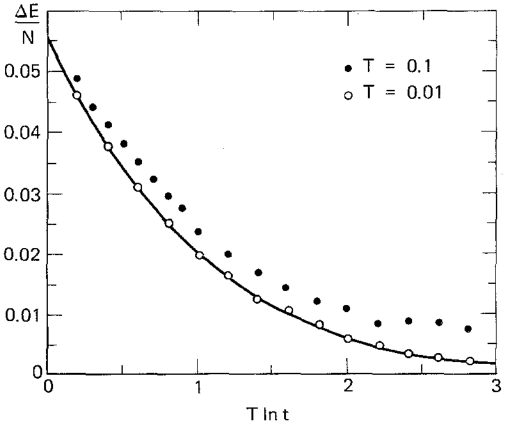

Fig. 4. Time-dependent energy of the Ising chain after quenched to a low temperature T. The data are calculated by assuming that even (or odd) states are in equilibrium in energy wells, Eq. (4.7). The solid line is a plot of Eq. (3.5).

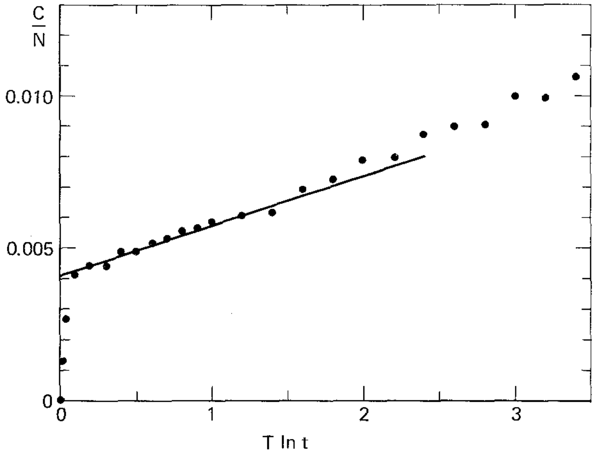

Fig. 5. The specific heat vs. $T \ln t$ at $T = 0.01$. Data points are the averages of 10^3 energy wells randomly generated. The solid line is for Eq. (4.16). The equilibrium value is $C_{eq}/N = 0.016$.

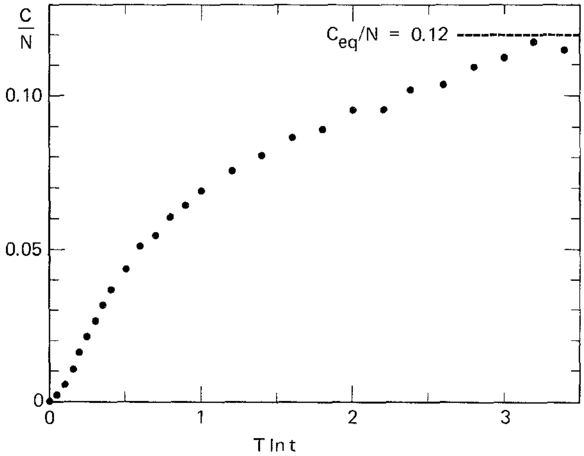

Fig. 6. The specific heat vs. $T \ln t$ at $T = 0.1$. Each point is an average of 10^3 energy wells. The equilibrium value is $C_{eq}/N = 0.12$.

in Section 7. For $T \ln t \ll 1$, we can use Eq. (4.12) to obtain

$$C(t)/N = \frac{\pi^2 T}{6}\left(\frac{1}{4} + \frac{\Lambda}{10} + \cdots\right) \tag{4.16}$$

This quantity comes from states with one kink in a well. For $T = 0.01$ Eq. (4.16), as plotted in solid line in Fig. 5, is in good agreement with the numerical calculation which includes states with any number of kinks in an energy well.

In Fig. 7 the time-dependent energy for $T = 0.1$ is compared with the Monte Carlo simulation results. In this figure each point (●) represents an average of 10^4 energy wells and the Monte Carlo data (+) are averaged over 50 independent systems of 10^3 spins. Our results are somewhat lower than Monte Carlo results. The agreements are within a few percent. We have tried to calculate the time-dependent specific heat of the random Ising chain by the Monte Carlo method, either by determining the fluctuation of the energy or by taking the derivative of $E(T, \Lambda)$ with respect to T. The uncertainties in the calculations are too large to obtain useful information for comparison with our theory.

It is important to note that to express Z_E and Z_O in terms of Z_+ and Z_- is of importance in practical calculations. Consider an energy well of 30 bonds, for example. If we calculate Z_E and Z_O by Eqs. (4.2) and (4.4),

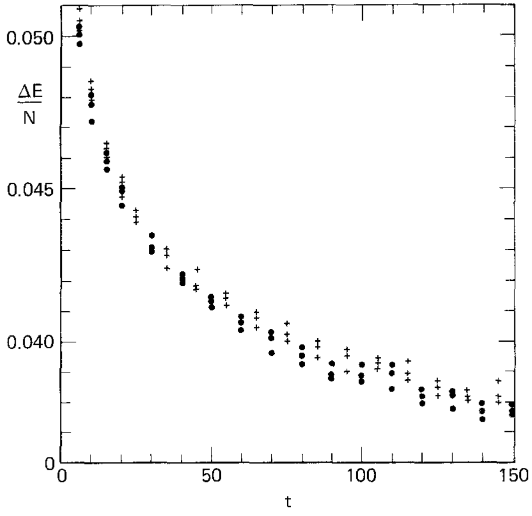

Fig. 7. The energy plotted vs. time. Each data point (●) is the average of 10^4 energy wells. The Monte Carlo data (+) are averages of 50 different systems.

about 10^2 multiplications need to be done [see Eq. (4.5)]. However, if we calculate $Z_{E,O}$ by Eqs. (4.1) and (4.3) directly, nearly $2^{30} \simeq 10^9$ terms must be summed up. For $\Lambda \gg 1$, most energy wells contain a large number of bonds. Equations (4.2) and (4.4) enable us to study the long time behavior numerically.

5. ZERO-FIELD SUSCEPTIBILITY

To study the time-dependent susceptibility we assume as above that all even states (or odd states) are in equilibrium in an energy well considered. For an energy well of n bonds it is convenient to rewrite the Hamiltonian as

$$H = -\frac{1}{2}\sum_{i=1}^{n} J_i S_i S_{i+1} - h\sum_{i=1}^{n+1} x_i S_i \tag{5.1}$$

where $x_i = x_{i-1}t_i$, and $S_i = x_i\sigma_i$. If the numbers of kinks at the energy bonds $J_1, J_2, \ldots$ are, respectively, $k_1, k_2, \ldots,$ where $k_i = 1$, or 0, then

$$S_i = S_1(-1)^{K_i} \qquad K_i = k_1 + k_2 + \cdots + k_{i-1} \tag{5.2}$$

For an energy well with even (or odd) states, the susceptibility is

$$\chi_{E,O} = \frac{1}{T}\left\langle\left(\sum_i x_i S_i\right)^2\right\rangle_{E,O} - \frac{1}{T}\left\langle\sum_i x_i S_i\right\rangle_{E,O}^2 \tag{5.3}$$

where $\langle\ \rangle_{E,O}$ means a thermal average over even (or odd) states. Since x_i as well as t_i are ± 1 randomly with equal probability, when we average over the distribution of t_i, $\langle x_i x_j\rangle = \delta_{ij}$. Therefore

$$\chi_{E,O} = \frac{1}{T}\sum_{j=1}^{n+1}\left(1 - \langle S_j\rangle_{E,O}^2\right) \tag{5.4}$$

Note that (i) $\langle S_j\rangle_{E,O}^2 = 1$ for $j = 1$ and $n + 1$, and (ii) if all even states and odd states are allowed to mix, all $\langle S_j\rangle = 0$ in the zero field limit.

5.1. Leading Term of the Susceptibility at Small $T\ln t$

As described in Section 4.2, when we average a thermodynamic quantity over the distribution of J, the leading contribution at low temperatures comes from the one-kink states. At small Λ, the leading contribution to χ also comes from one-kink states. If there is one kink at J_k of an energy well of n bonds, then $S_1 = S_2 = \cdots = S_k = -S_{k+1} = \cdots = -S_{n+1}$. The net magnetization of the well in the state is

$$m_k = \sum_{i=1}^{n+1} S_i x_i = S_1\left(\sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n+1} x_i\right) \tag{5.5}$$

Consider only the one-kink states; Eq. (5.3) gives

$$\chi_O = \frac{1}{2T} \frac{\sum_i \sum_j (m_i - m_j)^2 \exp^{[-(J_i + J_j)/T]} + \cdots}{\left[\sum_i \exp^{(-J_i/T)} + \cdots\right]^2} \tag{5.6}$$

$$= \frac{1}{T} \sum_{i \neq g} \frac{(m_i - m_g)^2 e^{-\Delta_i/T}}{\left(1 + e^{-\Delta_i/T}\right)^2} + \cdots \tag{5.7}$$

Here g denotes the state with a kink at the bottom of the energy well, and $\Delta_i = J_i - J_g$. Terms with more than one kind of Boltzmann factor are neglected.

The susceptibility, Eq. (5.7), depends on the position of J_g in the well. When we sum over all energy wells, all possible positions of J_g should be considered separately:

$$\frac{\chi}{N} = \frac{1}{2T} \sum_{n=2}^{\infty} \left[\sum_{g=1}^{n} \sum_{i=1}^{n} \left\langle (m_i - m_g)^2 \right\rangle_{\{x_i\}} \right]$$
$$\times e^{-2\Lambda} \int_0^{\infty} e^{-4J_g} dJ_g \left[\int_{J_g}^{J_g + \Lambda} p(J)\, dJ \right]^{n-2} \int_0^{\Lambda} \frac{e^{-\Delta} e^{-\Delta/T}}{\left(1 + e^{-\Delta/T}\right)^2} d\Delta \tag{5.8}$$

Here the first factor $1/2$ indicates that only one half of the wells are in odd states. $\langle\ \rangle_{\{x_i\}}$ is to be averaged over the signs of $\{x_i\}$ (or $\{t_i\}$). The integrations over J's are the same as before.

Since $x_i = \pm 1$ randomly, $\langle (m_i - m_g)^2 \rangle = 4|i - g|$, the summations over g and i give

$$\sum_{g=1}^{n} \sum_{i=1}^{n} \left\langle (m_i - m_g)^2 \right\rangle_{\{x_i\}} = \frac{8}{6} (n+1)(n)(n-1) \tag{5.9}$$

Equation (5.8) then reduces to

$$\frac{\chi}{N} = \frac{e^{-2\Lambda}}{T} \int_0^{\infty} \frac{e^{-4J_g} dJ_g}{\left[1 - e^{-J_g}\left(1 - e^{-\Lambda}\right)\right]^4} \int_0^{\Lambda} \operatorname{sech}^2 \frac{\Delta}{2T} e^{-\Delta} d\Delta \tag{5.10}$$

For small Λ, the first integration yields

$$I = \tfrac{1}{4} + \tfrac{4}{5}\left(1 - e^{-\Lambda}\right) + \cdots, \qquad \Lambda \ll 1 \tag{5.11}$$

and the second integration yields

$$I' = 2T \tanh\left(\frac{\Lambda}{2T}\right) + O(T^2), \qquad T \ll 1 \tag{5.12}$$

Therefore, for small Λ we obtain

$$\frac{\chi}{N} \simeq \left(\frac{1}{2} + \frac{3}{5}\Lambda + \cdots\right) \tanh\left(\frac{\ln t}{2}\right) \tag{5.13}$$

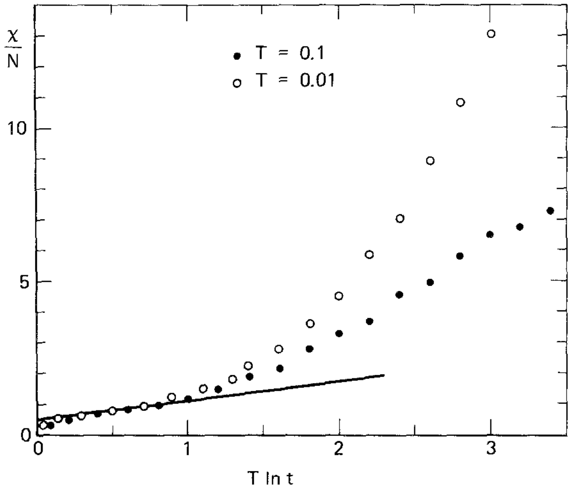

Fig. 8. Zero field susceptibility vs. $T \ln t$ for $T = 0.01$ and $T = 0.1$, respectively. Data are obtained by assuming that even (or odd) states are in equilibrium in energy wells considered. Each data point is an average of 10^3 wells. The solid line is the result when only one kink can exist in a well, Eq. (5.13).

At low temperatures, and for t not too small, there is a range of t such that $\Lambda \ll 1$ and $\ln t > 1$. Then, $\tanh(\ln t/2) \simeq 1$, and the susceptibility, $\chi/N \simeq \frac{1}{2} + \frac{3}{5}\Lambda$, does not explicitly depend on T (see Fig. 8).

5.2. Numerical Calculation of the Susceptibility

For general values of Λ, the time-dependent susceptibility of the Ising chain can be evaluated numerically by the method described in Section 4.3. We first generate a number of energy wells and for each well calculate χ_E and χ_O by Eq. (5.4). To calculate $\langle S_j \rangle_{E,O}$ numerically, it is convenient to write, apart from a constant factor ± 1,

$$\langle S_j \rangle_{E,O} = \frac{Z_+}{Z_+ \pm Z_-} \left[\prod_{i=1}^{j-1} \tanh \frac{J_i}{2T} \pm \prod_{i=j}^{n} \tanh \frac{J_i}{2T} \right] \tag{5.14}$$

where the plus sign is for even states and the minus sign for odd states, and $Z_\pm$ are given by Eq. (4.5). In deriving Eq. (5.14) we rewrite Eq. (5.2) as

$$S_j = S_1 \exp\left[i\pi (k_1 + k_2 + \cdots + k_{j-1}) \right] \tag{5.15}$$

For $T = 0.01$ and 0.1 the susceptibilities are shown in Fig. 8. Our numerical calculations also show that odd states contribute more to the time-dependent susceptibility than even states when $T \ln t < 1$. For $T \ln t \gg 1$ both even states and odd states contribute the same amount. Figure 9

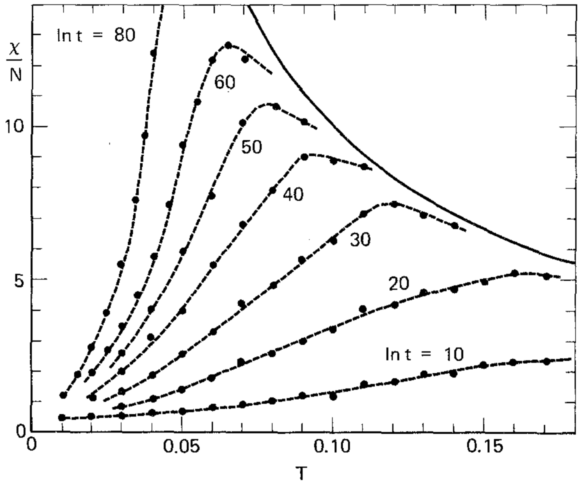

Fig. 9. Zero field susceptibility vs. temperature for various times. The solid line is the equilibrium susceptibility $\chi_{eq}/N = 1/T$.

shows the χ vs. T plots for different times. At each time, a hump at a certain temperature is observed.

6. THE EFFECT OF A MAGNETIC FIELD

All our discussion above has been on properties under zero or infinitesimal field. We now turn to the case of nonzero field h. Apparently, as soon as $h \neq 0$, the problem becomes far more difficult. We have not been able to carry out the corresponding simple analytic calculations in the previous sections. Below we present some heuristic arguments, which bring out some qualitative features of the model under $h \neq 0$.

We shall restrict our discussion to the small field case, i.e., $h \ll 1$; the field is much weaker than the typical spin–spin interaction J. The most important effect of the field is its large scale effect. However, we do not have a clear-cut classification of scales, but only rough estimates. We shall first present a heuristic derivation of some ground-state and low-temperature properties, and point out the new energy and length scales implied by h. Then we shall obtain similar results from a simple scaling argument.

6.1. Ground-State and Low-Temperature Properties

We find it convenient to use the spin variables S_i as given by Eq. (5.1) such that all the spin–spin interactions J_i are positive. In terms of S_i, the uniform magnetic field is turned into a random magnetic field with magnitude h and a random sign x_i, i.e., $h_i = x_i h$.

Let us first make a rough estimate of the ground-state energy. For $h = 0$, the answer is trivial, all S_i are the same, i.e., there is no kink.

When the field h is on, the picture changes drastically. Suppose the bonds J_n and J_m are l spacings apart. If all spins are up ($S_i = 1$), then the field energy for spins between bonds J_n and J_m is

$$-\sum_{i=n+1}^{m} h_i \equiv -\phi_{mn} \tag{6.1}$$

If l is large, then the magnitude of ϕ_{mn} is of the order

$$\phi_{mn} \sim h l^{1/2} \tag{6.2}$$

as a consequence of the central limit theorem. If ϕ_{nm} happens to be negative and

$$2|\phi_{mn}| > J_n + J_m \tag{6.3}$$

then the energy would be lowered by flipping all the spins between J_n and J_m. Two kinks are created, increasing the energy by $J_n + J_m$, but the field energy is lowered by $2|\phi_{mn}|$.

Now we can see roughly how to generate the ground state configuration. Let J_c be a small quantity. Now we locate all bonds $J_i < J_c$, i.e., locate all the weak bonds. The region between two successive weak bonds is a domain, of average size

$$l \sim J_c^{-(n+1)} \tag{6.4}$$

since $\int_0^{J_c} p(J)\, dJ \sim J_c^{n+1}$ is the number of weak bonds per unit length. Here we used Eq. (2.4) for $p(J)$. We then turn the spins in each domain to point along ϕ [see Eq. (6.1)], i.e., to follow the net field in the domain. If J_c is sufficiently small, the energy is lowered.

Now we increase J_c. Then smaller domains of spins can be turned to follow the net field in these small domains, the lower the energy further. This process is continued until J_c has been increased to become comparable to ϕ, i.e.,

$$J_c \sim h l^{1/2} \tag{6.5}$$

According to Eq. (6.4), this is the same as saying

$$J_c \sim h^{2/(n+3)} \tag{6.6}$$

$$l \sim h^{-2(n+1)/(n+3)} \equiv \xi \tag{6.7}$$

This ξ is an estimate of the smallest domain sizes. It is more instructive to think of smaller domains nested in larger domains [even though one can think of the system as made up of small domains of size ξ, Eq. (6.7)]. The larger the domain, the stronger the effect of the random field (i.e., $\phi \sim hl^{1/2}$).

The ground-state energy per spin can now be estimated from Eqs. (6.6) and (6.7).

$$[E(h) - E(0)]/N \sim h\xi^{1/2}/\xi$$
$$\sim h^{(2n+4)/(n+3)} \tag{6.8}$$

Note that this is not proportional to h^2 as one might have expected. We can get the magnetization M and the susceptibility by differentiation:

$$M/N \sim h^{(n+1)/(n+3)} \tag{6.9}$$

$$\chi/N \sim h^{-2/(n+3)} \tag{6.10}$$

Clearly, they are not analytical functions of h. The susceptibility diverges as $h \to 0$.

Next, let us assume a nonzero temperature T. Consider a domain of size l. The spins in the domain either line up along the net field in the domain or against it. One can easily write down the equilibrium properties of such a two-state system. The free energy is

$$f \sim -T \ln 2 \cosh(\epsilon/T) \tag{6.11}$$

where $\epsilon \sim hl^{1/2} - J$ with $J < J_c$. Since $J \sim hl^{1/2}$ most domains have $\epsilon \sim hl^{1/2}$. The magnetization $(-\partial f/\partial h)$ is

$$m \sim l^{1/2} \tanh(hl^{1/2}/T) \tag{6.12}$$

The magnetization per spin in this domain is m/l, and the total magnetization per spin of the Ising chain is thus

$$\frac{M}{N} \sim \xi^{-1/2} \tanh\left(\frac{h\xi^{1/2}}{T}\right) \tag{6.13}$$

Define the temperature T_h by

$$T_h \equiv h\xi^{1/2} \sim h^{2/(n+3)} \tag{6.14}$$

For $T \ll T_h$, Eq. (6.13) reduces to Eq. (6.9). And for $T \gg T_h$, we have $M/N \sim h/T$, as given by Eq. (2.12). In the later case, the cutoff ξ in Eq. (6.13) is not quite justified since domains smaller than ξ down to $1/T$ are involved. However, Eq. (6.13) is independent of the cutoff in this case.

The specific heat contributed by a domain, $-T\partial^2 f/\partial T^2$, is

$$c \sim \frac{\epsilon^2}{T^2} \operatorname{sech}^2 \frac{\epsilon}{T} \tag{6.15}$$

Even though the order of magnitude of ϵ is $hl^{1/2}$, here we must be more careful. The specific heat comes from those domains which flip, i.e., with $\epsilon \leqslant T$. Here ϵ is the kink energy (kinks are created or annihilated when a domain flips) minus the field energy. When these two energies nearly cancel, the domain would then contribute significantly to the specific heat and the entropy. We cannot simply write the specific heat per spin of the system as $C/N \approx c/\xi$. Since the number of weak bonds per spin is $\sim\int_0^{J_c} J^n \, dJ$, when we average c over the weak bonds, we obtain

$$\frac{C}{N} \sim \int_0^{J_c} J^n \, dJ \left(\frac{J - h\xi^{1/2}}{T} \right)^2 \operatorname{sech}^2 \left(\frac{J - h\xi^{1/2}}{T} \right) \tag{6.16}$$

At low temperatures $T \ll T_h \ll 1$, Eq. (6.16) reduces to

$$\begin{aligned} \frac{C}{N} &\sim \int_{-T_h}^{J_c - T_h} (\epsilon + T_h)^n \, d\epsilon \left(\frac{\epsilon}{T} \right)^2 \operatorname{sech}^2 \frac{\epsilon}{T} \\ &\sim \int_{-\infty}^{\infty} T(Tx + T_h)^n x^2 \operatorname{sech}^2 x \, dx \\ &\sim T_h^n T + O(T_h^{n-1} T^2) \end{aligned} \tag{6.17}$$

Equation (6.17) is different from Eq. (2.9), $C/N \sim T^{n+1}$, which is valid for $T_h \ll T \ll 1$.

The main result of the above analysis is the appearance of the new length scale $\xi = h^{-2(n+1)/(n+3)}$, and the energy (or temperature) scale $T_h \approx h^{2/(n+3)}$. When $T \ll T_h \ll 1$, the specific heat is $C \sim T_h^n T$, Eq. (6.17), and the susceptibility $\chi \sim 1/T_h$, Eq. (6.10). On the other hand, when $T_h \ll T \ll 1$, T_h in the above results should be replaced by T, i.e., the specific heat is $C \sim T^{n+1}$, Eq. (2.9), and the susceptibility $\chi \sim 1/T$, Eq. (2.12). Since $n > -1$, $T_h \gg h$ at small fields, there is a range of temperature between h and T_h: $h \ll T \ll T_h \ll 1$. Over this range, most of the domains are held fixed by the field. Only a small fraction of the domains can flip giving rise to the specific heat (and hence the entropy) of the order $T_h^n T$. It is so far too crude an estimate. Later in this section, we shall see that this result also follows from a scaling argument, and puts Eq. (6.17) on a firmer ground.

6.2. Approach to Equilibrium in a Field

When the system is cooled from a high temperature, with a field on, down to a very low temperature, the system will be "closer to equilibrium" than it would be without the field. In short, the "field-cooled" state is more like an equilibrium state than the "zero-field-cooled" state. This seems to be an experimental observation for many real spin glassses. It also appears to be the case for our model system, although we do not claim that this model system realistically describes the mechanism in the experimental spin glasses.

As shown in the previous sections, when the system is quenched without a field, kinks are randomly distributed. The kinks trapped in large energy wells can hardly get out. On the other hand, when a field is on, a large domain of size l will line up with the field provided that $hl^{1/2} \gg T'$, if T' is high enough for equilibrium to take place rapidly. When the system is cooled down from T' to T, this domain is already in the "right equilibrium configuration." Thus the metastable large energy wells appearing in the zero field case no longer appear in the field-cooled case. If T' is comparable to T_h, then the field needs to be strong enough to make ξ small enough in order that metastability is no longer present. In other words, if $T_h \sim hl^{1/2}$, then

$$l \sim \left(\frac{T_h}{h} \right)^2 = h^{-2(n+1)/(n+3)} = \xi \tag{6.18}$$

Thus, most of the relevant domains are in equilibrium if $T' \sim T_h$.

6.3. The Scaling Argument

The above discussion has been very crude. The classification of scales appears to be rather artificial. Here we shall use directly a simple scaling argument to obtain the main results given above.

The probability that a kink exists at a bond depends on the bond energy J as well as on the field h. If we assume that the probability f_k is a generalized homogeneous function then

$$f_k(J,h) = h^y K(J/h^x) \tag{6.19}$$

In the limit $J \ll h$, the probability to have a kink at J should be independent of h, and equal to $1/2$. This requires that $y = 0$, and $K(0) = 1/2$. The exponent x is to be determined.

From Eqs. (2.4) and (6.19) the kink energy per spin is

$$\begin{aligned} \frac{E_k}{N} &= \int_0^\infty J p(J) K\left(\frac{J}{h^x} \right) dJ \\ &= A h^{(n+2)x} K_{n+1} \end{aligned} \tag{6.20}$$

and the probability to have a kink at a given position is

$$\begin{aligned} \lambda &= \int_0^\infty p(J) K(J/h^x)\, dJ \\ &= A h^{(n+1)x} K_n \end{aligned} \tag{6.21}$$

where

$$K_n = \int_0^\infty z^n K(z)\,dz \tag{6.22}$$

The kinks divide the chain into sections of various lengths with average length $l_{av} = \lambda^{-1}$. In the low field limit the number of spins in a section of length l is much greater than 1. As there is no kink inside a section, the randomly oriented spins have a spin excess proportional to $\sqrt{l}$. Therefore the magnetization per site in this section is proportional to $\sqrt{l}/l = (l)^{-1/2}$. We can assume that the magnetization per spin of the Ising chain is proportional to $(l_{av})^{-1/2} = \lambda^{1/2}$, with (field-independent) proportional constant B, i.e.,

$$M/N = B\left[Ah^{(n+1)x}K_n\right]^{1/2} \tag{6.23}$$

The total energy per spin (zero-field ground state energy = 0) is the sum of the magnetic energy $-hM/N$ and the kink energy E_k/N,

$$E/N = Ah^{(n+2)x}K_{n+1} - Bh^{(x/2)(n+1)+1}(AK_n)^{1/2} \tag{6.24}$$

To minimize the free energy at small h we require that the power $(x/2)(n + 1) + 1$ be as small as possible, while the power of the positive term $(n + 2)x$ be not less than $(x/2)(n + 1) + 1$.This gives $x = 2/(n + 3)$, and the total energy

$$E/N = \left[AK_{n+1} - B(AK_n)^{1/2}\right]h^{(2n+4)/(n+3)} \tag{6.25}$$

To determine the function $K(z)$ we further require that the coefficient in Eq. (6.25) be as negative as possible, with the restriction that $K(z)$ is a decreasing function and $0 \leqslant K(z) \leqslant 1/2$. We have tried several functions K with adjustable parameters. It comes out that the lowest energy is obtained when $K(z) = 1/2$ for $z \leqslant [B^2(n + 1)/2A]^{1/(n+3)}$ and $= 0$ otherwise. The minimized energy is

$$\frac{E}{N} = -\frac{n+3}{2n+4}\left(\frac{B^2}{2}\right)^{(n+2)/(n+3)}\left(\frac{A}{n+1}\right)^{1/(n+3)} h^{(2n+4)/(n+3)} \tag{6.26}$$

and the magnetization $M = -\partial E/\partial h$ is

$$\frac{M}{N} = \left(\frac{B^2}{2}\right)^{(n+2)/(n+3)}\left(\frac{A}{n+1}\right)^{1/(n+3)} h^{(n+1)/(n+3)} \tag{6.27}$$

The constant B is difficult to determine even when $K(z)$ is known.

In the following we determine the coefficient B in a self-consistent way. Our calculation only provides an upper bound of the true ground-state energy. Assume that the energy bonds can have a kink with $f_k = 1/2$ for $J \leqslant J_c$, and no kink exists at bonds with $J > J_c$. The probability that a given bond has energy less than J_c is simply 2λ [λ is defined in Eq. (6.21)]. If we imagine that the Ising chain is divided into sections (or domains) by these low-lying energy bonds ($J \leqslant J_c$), the probability that a given section has length l is

$$p_l = 2\lambda(1 - 2\lambda)^{l-1} \tag{6.28}$$

For sections of l spins, the distribution of the spin excess, $m = \sum_{i=1}^{l} \sigma_i$, follows the Gaussian distribution for $l \gg 1$,

$$g(l, m) \simeq \left(\frac{1}{2\pi l}\right)^{1/2} \exp\left(\frac{-m^2}{2l}\right) \tag{6.29}$$

Therefore the magnitude of the spin excess has the average value $|m| = (2l/\pi)^{1/2}$ for sections of l spins.

When the magnetic field is turned on, if the spin excess of a section is parallel to h, nothing happens to the section. On the other hand, if the spin excess is antiparallel to h, we assume that all spins in the section reverse their directions. A kink occurs at the junction when one section reverses its spins and the other section does not. Exactly one half of the low-lying energy bonds have a kink. The magnetization is then

$$\frac{M}{N} = \frac{\sum_{l=1}^{\infty} (2l/\pi)^{1/2} (1 - 2\lambda)^{l-1} 2\lambda}{\sum_{l=1}^{\infty} l(1 - 2\lambda)^{l-1} 2\lambda} \simeq \sqrt{\lambda} \tag{6.30}$$

Here we have used

$$\begin{aligned} 1 + \sqrt{2}\,x + \sqrt{3}\,x^2 + \cdots &\simeq \int_0^\infty \sqrt{n}\, x^n \, dx \\ &\simeq \frac{\sqrt{\pi}}{2} (2\lambda)^{-3/2} \end{aligned} \tag{6.31}$$

for $x = 1 - 2\lambda$, and $\lambda \ll 1$. Equation (6.31) is correct in the low-field limit, where $\lambda \ll 1$. But Eq. (6.30) is only an approximation because many sections have spin excess much less than $\sqrt{2l/\pi}$. These sections may not behave as we assumed to reverse their spin directions.

Equation (6.30) gives the approximate value of the proportional constant in Eq. (6.23), $B = 1$. By setting $B = 1$ in Eqs. (6.26) and (6.27) we obtain an upper bound of the energy and a lower bound of the magnetization.

To check our result, we have calculated the magnetization of Ising chains of 10^4 spins, with $p(J) = \exp(-J)$, by the transfer matrix method. For the exponential distribution of J, $A = 1$ and $n = 0$ at small J. The M

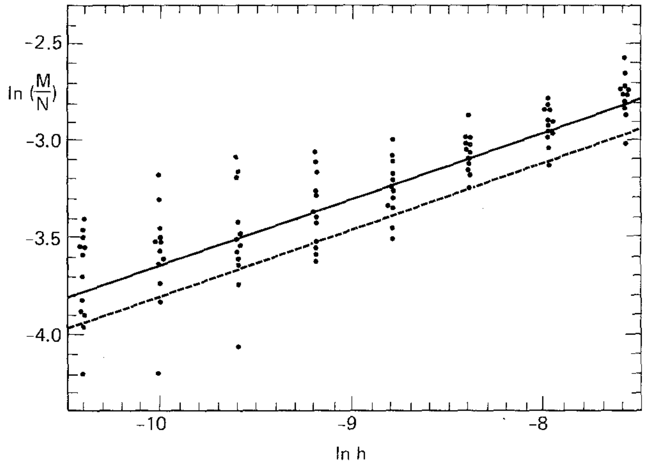

Fig. 10. Magnetization vs. external field in the logarithmic scale for 12 systems of 10^4 spins. Data are obtained by the transfer matrix method. The solid line represents the best fit $M/N = 0.74h^{1/3}$. The dashed line is from Eq. (6.27), $M/N = 0.63h^{1/3}$.

vs. h plot in logarithmic scale is shown in Fig. 10. In this figure, data of 12 different systems in the magnetic field from 3×10^{-5} to 5×10^{-4} are given. For a larger field, the $h^{1/3}$ power law is not obeyed exactly; and for smaller field, the fluctuation in M due to finite size of the system is comparable to the mean value of M.

The best fit to the data gives $M/N = 0.74h^{1/3}$. If we set $n = 0$, $A = 1$, and $B = 1$, Eq. (6.27) yields $M/N = 0.63h^{1/3}$. The exponent agrees very well. The coefficient obtained from Eq. (6.27) is 15% smaller than that obtained from the exact transfer matrix method. The agreement is not bad although we have made several approximations to arrive at Eq. (6.27). And $B = 1$ gives only a lower bound of the true magnetization.

The free energy per spin at a nonzero temperature can also be assumed to follow a scaling form

$$F(h, T) = -\frac{T}{\xi} f\left(\frac{h\xi^{1/2}}{T}\right) \tag{6.32}$$

where ξ is defined by Eq. (6.7). For $T \to 0$, we must have $f(x) \sim x$ for large x in order that T drops out. Thus we get back to the result Eq. (6.8). For large T and small h, we expect F to be analytic in h. The leading terms in $f(x)$ for small x, that make F analytic are $x^{-(n+1)}$, $x^{(1-n)/2}$, and x^2, since $\xi^{-1}(h\xi^{1/2})^{-(n+1)} = 1$, $\xi^{-1}(h\xi^{1/2})^{(1-n)/2} = h$ and $\xi^{-1}(h\xi^{1/2})^2 = h^2$. The term $x^{(1-n)/2}$ should vanish because F is an even function of h, thus we have for small field

$$F(h,T) \sim -T^{n+2} - \frac{h^2}{T} + \cdots \qquad \text{for} \quad T_h \ll T \ll 1 \tag{6.33}$$

Equation (6.33) predicts $M/N \sim h/T$, and $C/N \sim T^{n+1}$ as before.

The low-temperature ($T \ll T_h$) specific heat can be obtained by expanding $f(x)$ for large x:

$$f(x) \sim x + \text{const} + x^{-1} + \cdots \tag{6.34}$$

The constant term should be zero so that the entropy vanishes at $T = 0$. The specific heat per spin is

$$\begin{aligned} \frac{C}{N} &= -T\frac{\partial^2 F}{\partial T^2} \\ &\sim T\frac{\partial^2}{\partial T^2}\left(\frac{T^2}{h\xi^{3/2}}\right) \\ &\sim T_h^n T \end{aligned} \tag{6.35}$$

here $h\xi^{3/2} = h \cdot h^{-3(n+1)/(n+3)} = h^{-2n/(n+3)} = T_h^{-n}$. This reconfirms the result of Eq. (6.17).

It is interesting to note that when $n = 0$ (the Gaussian distribution, or the exponential distribution) the specific heat, and hence the entropy appears to be independent of h. Such approximate independence of h has been observed in model spin glass of long-range interaction with a Gaussian distribution of $p(J)$.[10]

7. CONCLUSIONS AND DISCUSSION

We have studied the low-temperature properties of the Ising chain with nearest-neighbor random interaction. At low temperatures, the equilibrium properties are determined mainly by the distribution of bond energies with $J \ll 1$, where $p(J) \approx AJ^n$. The leading contribution to the thermal energy is proportional to AT^{n+2}. This energy is contributed by kinks at the energy minima alone. At zero temperature and in a small field, we have found that the magnetization is proportional to $h^{n+1/n+3}$. For $n = 0$, the exponent $1/3$ agrees with that predicted from exact transfer matrix calculation for Ising chains randomly generated according to the distribution $p(J) = e^{-J}$.

The dynamic properties of the system depend on the whole range of the distribution function $p(J)$. Especially the long time behaviors depend heavily on the distribution at $J \gg 1$. Our initial condition (set $t = 1$) is the state that kinks exist only at energy minima with probability $1/2$. This state can be obtained approximately at low temperatures when the system is suddenly quenched from high temperatures, or when a strong magnetic field is suddenly switched off. After a short period of fast relaxation, all kinks move down to the energy minima, and annihilate in pairs.

Starting from our initial state, the kinks at the bottom of the energy wells can climb over the energy barrier and move to other wells. The time needed for a kink to climb over a barrier of height Λ is about $t \sim e^{\Lambda/T}$. Therefore at time t, energy wells with barriers lower than $\Lambda = T \ln t$ need not be considered. At extremely low temperatures we can assume that at most one kink can exist in a well. A crude result for the relaxation of the thermal energy and the remanent magnetization is obtained. The magnetization relaxes much slower than the thermal energy. This is qualitatively in agreement with the Monte Carlo data. Our crude theory, however, predicts faster relaxation than the Monte Carlo simulation both for the thermal energy and for the remanent magnetization.

For nonequilibrium systems, the energy and the magnetization are well defined quantities; but their derivatives are not. The energy wells are defined in terms of minimum barrier height $\Lambda = T \ln t$, which is temperature dependent. When we want to calculate the specific heat, for example, we need to specify how it is defined. If we defined it in terms of local energy fluctuations at a given temperature, we want to keep Λ fixed, i.e., $C = \partial E(T, \Lambda)/\partial T$. If we define it in terms of measurements of energy changes carried out at different temperatures, then we must consider the change in Λ, and define $C = \partial E(T, t)/\partial T$.

At low temperatures, and for $T \ln t < 1$, the time-dependent thermodynamic properties are contributed mainly from the odd states; and they vary with time logarithmically, i.e., $A(t) = A(0) \pm BT \ln t$, for general distribution $p(J)$. At $T \ln t \gg 1$, the even states and the odd states contribute the same amount to each thermodynamic quantity. The time-dependent thermodynamic quantities approach their equilibrium value very slowly. The thermalization depends on the distribution $p(J)$ at large J. For example, the thermal energy decays approximately as $p^2(\Lambda)$, from $\Lambda \gg 1$, for the exponential, or the Gaussian distribution.

In this paper we have assumed that the time required for a kink to climb over a barrier of height Λ is $t = \tau e^{\Lambda/T}$. Here $\tau = 1$ is the time required for a kink to move to a nearest-neighbor lower energy bond. Since all energy wells with the same barrier height may have very different structure, the time required for a kink to move out of an energy well should depend on the details of the energy bonds in the well, such as the number of bonds in the well, the position of the bottom, etc. We have not taken this into consideration. It is very difficult to do so. An important feature is that for large Λ, the average number of bonds in the energy well is also large. The time required for a kink to move out of the energy well should be $\sim e^{\Lambda/T} \times$ (number of bonds), rather than $e^{\Lambda/T}$. This is probably the main reason that our theory predicts a faster relaxation of the energy and magnetization than the Monte Carlo results.

When the magnetic field is turned on, somehow the mathematical problem becomes much more difficult. Nevertheless, we have drawn a few interesting conclusions of plausible arguments. Perhaps the most important conclusion is that the field has a strong effect at large scales and it eliminates the metastability of large energy wells. Also important is the new temperature scale $T_h = h^{2/(n+3)}$. When $T_h \ll T \ll 1$, the specific heat and the susceptibility are nearly field independent. On the other hand, when $T \ll T_h \ll 1$, the specific heat is proportional to $T_h^n T$, and the susceptibility is temperature independent, $\sim 1/T_h$. For the case $n = 0$, i.e., $p(0) \neq 0$, e.g., the Gaussian, or the exponential distribution of $p(J)$, the specific heat, and hence the entropy is approximately independent of h for the entire low-temperature range.

The properties of this model do bear some resemblance to those of spin glasses. The major problem in understanding spin glass would be the enumeration of metastable states in a straightforward way. The above analysis shows that the enumeration can be very different from the independent cluster approximation. It also shows that we are quite far from being able to study general metastable states effectively.

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